DYNAMICS OF STRUCTURES

Dynamic Response Factors

- The steady state harmonic response is:

$$u(t)=\frac{p_o}{k}\frac{1}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\sin(\Omega t-\phi)$$

$$\frac{u(t)}{p_o/k}=R_d\sin(\Omega t-\phi)$$

$$R_d=\frac{1}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}$$

$$\frac{\dot{u}(t)}{p_o/\sqrt{km}}=\sqrt{\frac{m}{k}}R_d\Omega\cos(\Omega t-\phi)=R_v\cos(\Omega t-\phi)$$

$$\frac{\ddot{u}(t)}{p_o/m}=-\frac{m}{k}R_d\Omega^2\sin(\Omega t-\phi)=-R_a\sin(\Omega t-\phi)$$

$$\boxed{\frac{R_a}{\beta}=R_v=\beta R_d}$$

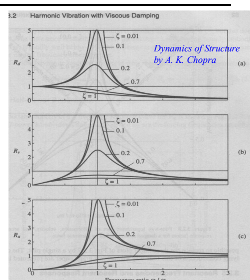
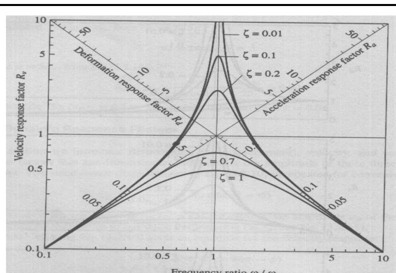


Figure 3.2.7 Deformation, velocity, and acceleration response factors for a damped system excited by harmonic force.

Four-way Logarithmic Plot



Dynamics of Structure
by A. K. Chopra

Figure 3.2.8 Four-way logarithmic plot of deformation, velocity, and acceleration response factors for a damped system excited by harmonic force.

Resonant Frequencies and Responses

- The frequency at which largest response amplitude occurs is called *resonant frequency*
- The steady state displacement, velocity and acceleration response of a dynamic system are:

$$u(t)=\frac{p_o}{k}\frac{1}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\sin(\Omega t-\phi),\qquad \dot{u}(t)=\frac{p_o}{k}\frac{\Omega}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\cos(\Omega t-\phi)$$

$$\ddot{u}(t)=-\frac{p_o}{k}\frac{\Omega^2}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\sin(\Omega t-\phi),$$

- By setting the first derivatives of u, $\dot{u}$ and $\ddot{u}$ w.r.t β equal to zero we can determine the resonant frequencies.
  - Displacement resonant frequency ratio : $\beta_d=\sqrt{1-2\zeta^2}$
  - Velocity resonant frequency ratio : $\beta_v=1$
  - Acceleration resonant frequency ratio : $\beta_a=1/\sqrt{1-2\zeta^2}$

Resonant Frequencies and Responses (Cont..)

- The three dynamic response factors are:

$$R_d=\frac{1}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\quad\Rightarrow R_d=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$R_v=\frac{\beta}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\quad\Rightarrow R_v=\frac{1}{2\zeta}$$

$$R_a=\frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\quad\Rightarrow R_a=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Half-Power Bandwidth

- The difference of frequency ratios on either side of the resonant frequency at which the amplitude is $1/\sqrt{2}$ times the resonant amplitude. For small damping:

$$R_d=\frac{1}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}},\qquad R_{d,resonant}=\frac{1}{2\zeta}$$

- For half-power bandwidth:

$$\frac{1}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}=\frac{1}{\sqrt{2}}\frac{1}{2\zeta}\Rightarrow\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2=8\zeta^2$$

$$\left(\beta^2\right)^2-2\left(1-2\zeta^2\right)\left(\beta^2\right)+1-8\zeta^2=0$$

$$\beta^2=\left(1-2\zeta^2\right)\pm\sqrt{\left(1-2\zeta^2\right)^2-\left(1-8\zeta^2\right)}=\left(1-2\zeta^2\right)\pm 2\zeta\sqrt{1+\zeta^2}$$

- For small damping:

$$\beta_a^2=\left(1-2\zeta^2\right)-2\zeta\ \_and\_\ \beta_b^2=\left(1-2\zeta^2\right)+2\zeta$$

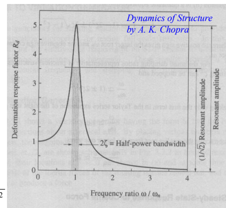
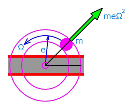


Figure 3.2.9 Definition of half-power bandwidth.

$$\beta_b-\beta_a=2\zeta\Rightarrow\zeta=\frac{\beta_b-\beta_a}{2}$$

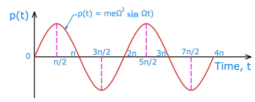
Rotating Machinery/Eccentric Mass Vibrator

- In case of rotating machinery:
  $p(t) = me\Omega^2 \sin \Omega t$
- Here p_o is $me\Omega^2$ the response is therefore:



$$u(t) = \frac{me\Omega^2}{k} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} \sin(\Omega t - \phi),$$

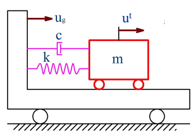
$$u(t) = \frac{e\beta^2}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} \sin(\Omega t - \phi),$$



Transmissibility: Support Movement

- Let the support move according to the equation:

$$u_g = G \sin \Omega t \quad \Rightarrow \quad \ddot{u}_g = -G\Omega^2 \sin \Omega t$$



- Equation of motion is:

$$m\ddot{u}^t + c\dot{u} + ku = 0 \quad \Rightarrow \quad m(\ddot{u} + \ddot{u}_g) + c\dot{u} + ku = 0$$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g = mG\Omega^2 \sin \Omega t$$

- Here $p(t) = mG\Omega^2 \sin \Omega t$
- Therefore $u(t) = \frac{mG\Omega^2}{k} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} \sin(\Omega t - \phi)$

$$\Rightarrow \quad u(t) = G \frac{\beta^2}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} \sin(\Omega t - \phi), \qquad \tan\phi = \frac{2\zeta\beta}{1-\beta^2}$$

Transmissibility: Support Movement (Cont..)

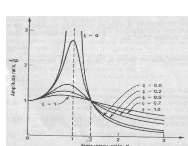
- The total displacement is:

$$u^t = u_g + u = G \sin \Omega t + G \frac{\beta^2}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} \sin(\Omega t - \phi)$$

$$\Rightarrow u^t = \chi \sin(\Omega t - \gamma), \quad \text{where} \quad \chi = G\sqrt{\frac{1+(2\zeta\beta)^2}{(1-\beta^2)^2 + (2\zeta\beta)^2}}, \quad \text{and} \quad \tan\gamma = \frac{2\zeta\beta^3}{(1-\beta^2)^2 + (2\zeta\beta)^2}$$

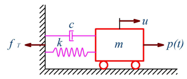
- The ratio of amplitude of total displacement (χ) to amplitude of ground displacement (G) is called transmissibility (TR)

$$TR = \frac{\chi}{G} = \sqrt{\frac{1+(2\zeta\beta)^2}{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$



Force Transmission

- The total force transmitted to the base is:
  $f_T = f_D + f_S = c\dot{u} + ku$



- The displacement and velocity response to harmonic force is:

$$u(t) = \frac{p_o}{k}\frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}}\sin(\Omega t - \phi), \qquad \dot{u}(t) = \frac{p_o}{k}\frac{\Omega}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}}\cos(\Omega t - \phi)$$

- Therefore:

$$f_T = p_o \frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}}\left\{\sin(\Omega t - \phi) + \frac{c}{k}\Omega\cos(\Omega t - \phi)\right\}$$

$$\Rightarrow f_T = p_o\sqrt{\frac{1+\left(2\zeta\beta\right)^2}{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}}\sin(\Omega t - \gamma)$$

- Transmissibility is also the ratio amplitudes of force transferred to the base to the applied force

$$TR = \frac{f_T}{p_o} = \sqrt{\frac{1+\left(2\zeta\beta\right)^2}{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}}$$